Technical Comments

AIAA 82-4310

Comment on "Expanded Third-Order Markov Undulation Model"

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IN Ref. 1, the author claims to have synthesized a nine-state Gauss-Markov model that yields the auto and cross correlation functions of nine quantities (a total of 81 functions). A one-dimensional process cannot, in general, be represented by a one state model. (For example, a third-order Markov process requires three states to model.) If the author had succeeded in capturing a nine-dimensional process with a nine state system it would be a remarkable result. Unfortunately, he has not.

The author states in Eq. (27) that the dynamics matrix can be evaluated by

$$F=\dot{\phi}^T(0)\phi(0)^{-1} \tag{1}$$

where $\phi(\tau)$ is a given 9×9 matrix of correlation functions.

$$\phi(\tau)=E[X(t)X^T(t+\tau)] \tag{2}$$

No where in the argument leading up to Eq. (1) were specific properties of $\phi(\tau)$ invoked, and in his conclusions Gerber states that Eq. (1) provides a "systematic route" for developing "multidimensional Markov models for other problems." This conclusion cannot be valid! The F matrix depends only on the value and derivative of $\phi(\tau)$ at zero. Changing the correlation functions, but leaving the values and slopes at zero the same, has no effect on F, but clearly has an effect on $\phi(\tau)$.

The mistake occurs just prior to Eq. (27). Equation (25) can be manipulated to

$$F=\dot{\phi}(\tau)^T\phi(\tau)^{-1} \tag{3}$$

If the quantity on the right-hand side is independent of τ, then Eq. (1) follows. If, however, the right-hand side is not constant, a system of the form considered by Gerber does not exist. Gerber neglected to examine whether the quantity $\dot{\phi}(\tau)^T\phi(\tau)^{-1}$ is independent of τ for his particular $\phi(\tau)$. The quantity is not independent of τ, and his system does not do what is claimed.

A method for synthesizing a state-space model to realize a given covariance function matrix can be found in Ref. 2.

References

[1] Gerber, M.A., "Expanded Third-Order Markov Undulation Model," *Journal of Guidance and Control,* Vol. 4, Sept.-Oct. 1981, pp. 495-501.

[2] Koehler, G.F., "Synthesizing State-Space Models to Realize Given Covariance Functions," *Proceedings of AIAA Guidance and Control Conference,* pp. 131-137, Albuquerque, New Mex., Aug. 1981; submitted for publication to *Journal of Guidance and Control.*

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Comment on "Expanded Third-Order Markov Undulation Model"

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Introduction

REFERENCE 1 lists three major objectives: to extend the third-order Markov undulation model of Ref. 2 by evaluating the auto and cross correlation functions corresponding to the elements of the anomalous gravity gradient tensor; to provide an improved state-space approximation to the Bessel functions which express certain cross correlations between anomalous gravity quantities; and to present a method for state-space realization of a matrix of correlation functions.

A systematic approach is outlined in Ref. 1 for obtaining gradient correlation functions of all orders starting from assumed correlation functions for the gravity anomaly and undulation of the geoid. However, an unfortunate choice for the undulation-anomaly cross correlation function invalidates many of the third-order Markov model functions derived in Ref. 1. The difficulties with this cross correlation are that it is statistically inconsistent with the related autocorrelations and structurally impossible for a state-space model. In addition, the procedure developed in Ref. 1 for realizing a state-space shaping filter model from these correlation functions is flawed. The problem with the filter synthesis method, as shown in the following, is that it is not sufficient to assure useful solutions. The filter parameters given provide only an approximate fit to the desired model statistics. No provision is made to control or monitor desirable properties such as goodness of fit and stability of the derived shaping filters.

Statistical Inconsistency

For a matrix of functions to be a correlation function of a vector random process, the corresponding composite covariance matrix of all random variables associated with sampling the vector process at any arbitrary, but finite, number of times must be positive semidefinite. This is a strong condition and can be somewhat tedious to check. A more convenient, although equivalent, requirement is that the corresponding spectral density matrix be positive semidefinite at every frequency. One necessary test is that the spectral coherence, given by

$$c_{ij}^2(\omega)=\frac{|\Phi_{ij}(\omega)|^2}{\Phi_{ii}(\omega)\Phi_{jj}(\omega)} \tag{1}$$

where Φ_{ij} is an auto $(i=j)$ or cross $(i\neq j)$ spectral density, must be less than unity at every frequency. By computing the appropriate Fourier transforms of Eqs. (1), (7), and (8) of Ref. 1, it is readily shown that the proposed spectral coherence between the undulation and gravity anomaly is

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given by

$$c^2_{N\Delta g}(\omega) = \frac{16d'^2(1+\omega^2 d^2)^6}{9d^2(1+5\omega^2 d^2)(1+\omega^2 d'^2)^6} \tag{2}$$

where $d' = 2d/3$. The value of Eq. (2) is plotted in Fig. 1. It is seen that the coherence exceeds unity over the entire band of frequencies

$$\frac{1.122}{d} < \omega < \frac{3.352}{d} \tag{3}$$

Thus, the proposed new cross correlation function is inconsistent with the autocorrelation functions.

In principle this defect could be corrected by multiplying the proposed cross correlation function by a suitable attenuation factor less than 0.78. However, this would result in a poor approximation to the desired Bessel function cross correlation formula. Furthermore, it would not solve a model structural problem as is now discussed.

Mixed Modes and State-Space Realization

A further difficulty with the proposed model results from the need to realize the specified functions as a state-space shaping filter. The autocorrelations of both geoid undulations and gravity anomalies are composed of the same third-order dynamic mode (i.e., system pole). The proposed cross correlation between them is from a different dynamic mode (also third order). By considering the spectral density matrix of a steady-state linear shaping filter, it can be shown that such a proposed realization is structurally impossible.

The spectral density matrix is given by

$$\Phi(\omega) = H(j\omega)QH^T(-j\omega) \tag{4}$$

where H is the system transfer function matrix and Q is the $(n \times n)$ input spectral density matrix (Q is assumed diagonal without loss of generality). From Eq. (4), it is readily shown that the ijth element of the spectral density is

$$\Phi_{ij}(\omega) = \sum_{k=1}^{n} q_{kk} h_{ik}(j\omega) h_{jk}(-j\omega) \tag{5}$$

For a dynamic mode to appear in Φ_{ij} it must be present either in h_{ik} or h_{jk} for some k. But then the same mode appears in Φ_{ii} or Φ_{jj} where it cannot be cancelled by addition because all terms of Eq. (5) are positive for $i=j$. Thus, any dynamic mode which appears in a cross correlation function must appear in at least one of the corresponding autocorrelation functions. The functions given in Ref. 1 are, therefore, not realizable by any linear shaping filter. Recent practice in the statistical gravity modeling community is to allow a better approximation to $\Phi_{N\Delta g}$ by approximating Φ_{NN} and $\Phi_{\Delta g \Delta g}$ as well (e.g., Ref. 3).

Realization Methodology

In view of the difficulties just illustrated, it is evident that the P, F, and Q matrices proposed in Ref. 1 cannot synthesize the desired statistics. In fact, because formulas (27) and (30) of Ref. 1 are based on necessary but not sufficient conditions, the synthesis is incomplete, as illustrated in Figs. 2 and 3. Note that the value and slope of each function are matched at zero shift. Because some states are related by integrals, reasonable accuracy is also obtained at increasing shift distances for certain states, e.g., the undulation. However, all of the realized functions are approximate, and in some cases the approximations break down quite quickly.

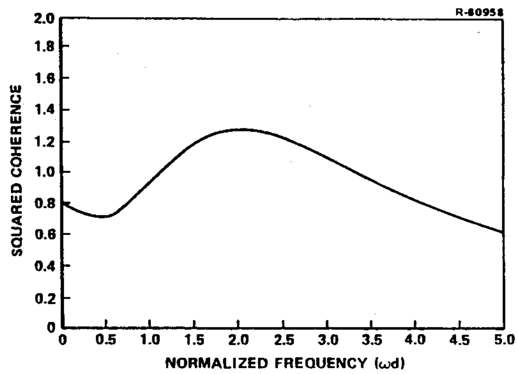

Fig. 1 Undulation-anomaly cross coherence corresponding to Eqs. (1), (7), and (8) of Ref. 1.

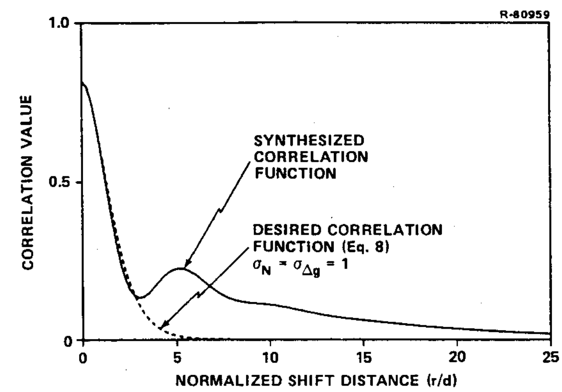

Fig. 2 Gravity anomaly-undulation cross correlation function.

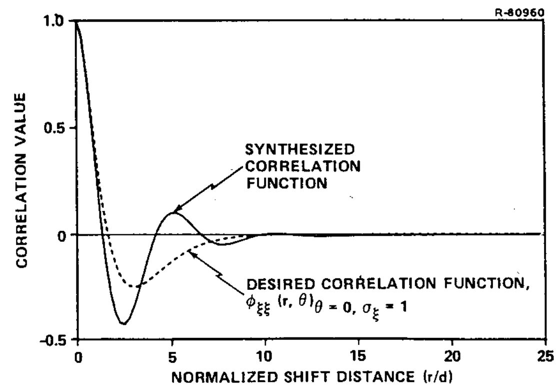

Fig. 3 Along-track deflection autocorrelation function.

The failure of the proposed realization procedure to give some indication of the problem in the original correlation function set is serious. The method will, without indication that anything is amiss, construct apparent solutions where none exist. An alternative synthesis deserves mention. In Ref. 3, it was found that the method given in Ref. 1 produced a Q matrix with diagonal elements zero. To overcome this problem Ref. 3 introduced a set of new, unspecified parameters and utilized a numerical optimization method to select values which cause the shaping filter to approximate the desired correlation functions.

Conclusion

It has been shown that the gravity model formulas presented in Ref. 1 are statistically inconsistent and cannot be fitted exactly by any shaping filter. The shaping filter presented therein results in synthesized functions which seriously diverge from the desired correlation functions at moderate shift distances.

References

[1] Gerber, M.A., "Expanded Third-Order Markov Undulation Model," *Journal of Guidance and Control,* Vol. 4, Sept.-Oct. 1981, pp. 495-501.

[2] Jordan, S.K., "Self-Consistent Statistical Models for the Gravity Anomaly, Vertical Deflections, and Undulation of the Geoid," *Journal of Geophysical Research,* Vol. 77, July 1972, pp. 3660-3670.

[3] Jordan, S.K., Moonan, P.J., and Weiss, J.D., "State Space Models of Gravity Disturbance Gradients," *IEEE Transactions on Aerospace and Electronic Systems,* Vol. AES-17, Sept. 1981, pp. 610-619.

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Reply by Author to S.L. Baumgartner, W.G. Heller, and G.F. Koehler

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BAUMGARTNER and Heller are correct in that the model presented does not exactly fit the correlation functions. The exact and modeled functions agree closely on rms values and correlation distances, but fail to match as well only at longer shift distances. Thus, the basic character of the gravity uncertainties is represented properly. The importance of the mismatch at longer shift distances depends on the desired use of the model.

Koehler's recommendation to use the canonical variate decomposition approach is a good one. As presented, the state-space model employed a force fit matching only the rms values and slopes of the correlation functions at zero shift. Legitimate concern over this was expressed by all three commenters over the possible consequences. If the canonical variate decomposition approach works as claimed, it provides a means to find the best model approximation for a given number of states and shows how adding more states may improve the model.

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